Received: *August 3, 2000*, Accepted: *September 13, 2000*

Nielsen-Olesen vortices in noncommutative abelian Higgs model

Dileep P. Jatkar[ab], **Gautam Mandal**[bc] **and Spenta R. Wadia**[c]

[a] *Mehta Research Institute of Mathematics and Mathematical Physics*
Chhatnag Road, Jhusi, Allahabad 211 019, India
[b] *Theory Division, CERN*
CH-1211, Geneva, 23, Switzerland
[b] *Department of Theoretical Physics, Tata Institute of Fundamental Research*
Homi Bhabha Road, Mumbai 400 005, India
E-mail: dileep@mri.ernet.inz, gautam.mandal@cern.ch, wadia@tifr.res.in

Abstract: We construct Nielsen-Olesen vortex solution in the noncommutative abelian Higgs model. We derive the quantized topological flux of the vortex solution. We find that the flux is integral by explicit computation in the large-θ limit as well as in the small-θ limit. In the context of a tachyon vortex on the brane-antibrane system we demonstrate that it is this topological charge that gives rise to the RR charge of the resulting BPS D-brane. We also consider the left-right-symmetric gauge theory which does not have a commutative limit and construct an exact vortex solution in it.

Keywords: Solitons Monopoles and Instantons, Brane Dynamics in Gauge Theories, Non-Commutative Geometry.

JHEP09(2000)018

Contents

1. Introduction and summary

Noncommutative field theories appear naturally in the low energy description of string theory in a constant Neveu-Schwarz antisymmetric tensor background [1]–[11]. They have also appeared previously in the study of $c = 1$ matrix model (two-dimensional string theory) [12] and of two-dimensional QCD [13]. In [2] it was observed that D-branes compactified on a torus with constant Neveu-Schwarz B field background gives rise to an effective noncommutative field theory on the compactified world-volume. There has been some study of perturbative dynamics of these theories [14]. However, their utility in understanding nonperturbative aspects of field theories has attracted more attention lately [15]. Presence of constant Neveu-Schwarz antisymmetric tensor field background tangential to the D-brane leads to noncommutative gauge theory on the D-brane world volume. In case of the abelian gauge theory it has been shown that the abelian noncommutative gauge theory is related by field redefinition to the Born-Infeld electrodynamics action on the D-brane [4]. Noncommutative gauge theories have been further discussed in [16, 17]; ref. [17] discusses monopole solutions in noncommutative U(1) gauge theory in $3+1$ dimensions.

The solitons in noncommutative scalar field theories studied in [15] are non-topological, and are stable at large θ. These solutions lack stability in the commutative limit as they violate Derrick's theorem. However, coupling of these scalars to, say, gauge fields could add stability to these solutions. Simplicity of the noncommutative field theory formalism in the large-θ limit was exploited in [18, 19] to gain better understanding of the tachyon condensation phenomenon in the non-BPS D-branes [20] using the physics of noncommutative solitons.

It is important to generalize the class of noncommutative solitons to accommodate solitonic solutions which have topological conserved charge. It is clear from the work of [18]–[22] that the topological solitons are relevant to the study of brane descent relations in the non-BPS brane dynamics. It has been conjectured that the topological charge of the tachyonic soliton on the brane worldvolume of the non-BPS Dp brane is the RR charge of the D$(p-k)$ brane [20, 22], where $k < p$ and for $k = 1$ the soliton is a kink, for $k = 2$, it is a vortex and so on. Study of topology in noncommutative field theories, however, is also important in its own right.

Here we will address this question in the abelian Higgs model. We will show that the noncommutative abelian Higgs model supports topological vortex solution with quantized flux, which can be any arbitrary integer. The model that we will study first is a left module on the Hilbert space, that is, in the complex Higgs field notation, the gauge field multiplies the Higgs field ϕ from the left whereas for $\bar{\phi}$ it multiplies from right. In section 2, we will briefly review the Nielsen-Olesen vortex solution in the ordinary abelian Higgs model, highlighting the first order, i.e., Bogomolnyi formulation.

We generalise the Bogomolnyi equations to the noncommutative abelian Higgs model in section 3 and obtain exact solutions in the large θ limit. In this limit it is possible to do a systematic $1/\theta$ expansion and we obtain corrections to the solution of the leading order equations of motion. We find that these corrections converge quite rapidly. In the large-distance limit this solution matches with the large distance Nielsen-Olesen ansatz for vortex solution in the ordinary abelian Higgs model. We also derive the Seiberg-Witten(SW) map for the noncommutative abelian Higgs model. We find that to order θ, the SW map for the gauge field A and the gauge transformation parameter λ is unaltered whereas the SW map for the Higgs ϕ is linear in ϕ. It has been argued in the pure gauge theory case [4], that the SW map in the zero slope limit gives a field redefinition from an ordinary field theory with the Born-Infeld action to the noncommutative gauge theory action which is quadratic in the gauge field strength. The SW map for A is a nonlinear function of A, whereas, as mentioned above, SW map for ϕ is linear in ϕ. Therefore it is tempting to conjecture

JHEP09(2000)018

that the corresponding action for the Higgs field in the ordinary field theory will retain its form with each term in the Higgs lagrangian multiplied by some function of the gauge field A.

We also determine the profile of the magnetic field B due to this vortex solution. In the original coordinates it is just a δ-function. This is as expected since in the large-θ limit, the dispersion generated by the derivative terms in the kinetic energy is totally suppressed. In the scaled coordinates, however, the magnetic field profile is proportional to the ground state wavefunction of the harmonic oscillator. The coefficient in front of this wavefunction encodes the topological charge of the solution.

In section 4, we discuss the issue of the topology of this solution. We show that this solution carries the topological conserved charge (magnetic flux) which is determined by the behaviour of the Higgs field. This topological charge is conserved, quantized and takes integer values. We establish this result in both small as well as large-θ limit. In the small-θ limit the leading result reproduces the integer vortex charge of the commutative limit; we show that higher order terms in θ expansion are total derivatives and fall too rapidly at large distances to contribute to the charge. In the large-θ limit too, the leading term itself gives the entire charge and subleading terms in the $1/\theta$ expansion do not contribute. We discuss the connection of this topological charge with various other quantized charges and indices. We also discuss the "semiclassical" limit of the field configuration in which the topology reduces to that of the commutative limit.

In section 5, we address the same question but for the left-right symmetric module. In this case we get an interesting vortex solution which at the face value seems to have charge 1 (using Witten's identification [21] of the vortex charge with Atiyah-Singer index of the Higgs field). The solution is not square integrable, however; in the operator language the trace of the topological charge over the one particle Hilbert space diverges and requires regularisation. The question of finding a consistent regularisation scheme to compute this charge is left for the future.

2. Ordinary abelian Higgs model

In this section we will briefly review the Nielsen-Olesen vortex solution to the abelian Higgs model [23], emphasising the Bogomolnyi limit of this model [24]. In the next section we will generalise these equations to the noncommutative abelian Higgs model and work only in the Bogomolnyi limit.

Let us start with the usual abelian Higgs model. The lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}|(\partial_\mu + ieV_\mu)\Psi|^2 - \lambda(|\Psi|^2 - |\Psi_0|^2)^2 . \tag{2.1}$$

As mentioned earlier we will concentrate on the Bogomolnyi limit. Bogomolnyi limit and the corresponding first order equations of motion are obtained from the energy functional by writing it in terms of strictly positive quantities and a topological charge density and then minimising the energy functional. We will take the static ansatz. That means we will set $V_0 = 0$ and $\partial_t V_m = 0 = \partial_t \phi$. Energy functional of the abelian Higgs model for such an ansatz is

$$E = \int d^2x \left[\frac{1}{4} F_{mn} F^{mn} + |D_m \Psi|^2 + \lambda (|\Psi|^2 - \Psi_0^2)^2 \right] . \tag{2.2}$$

Let us rescale coordinates and fields and write them down in terms of the following dimensionless variables

$$\begin{aligned} \Psi &= \Psi_0 \phi \,, & V_m &= \Psi_0 A_m \,, \\ x_m &= \frac{1}{e\Psi_0} y_m \,, & z &= \frac{1}{\sqrt{2}}(y_1 + i y_2) \equiv r\, e^{i\varphi} \qquad \text{and} \qquad E = 2\pi \Psi_0^2 \mathcal{E} \,. \end{aligned} \tag{2.3}$$

The Nielsen-Olesen ansatz for the asymptotic form of a vortex solution with winding number n is

$$\begin{aligned} \phi &= \exp(in\varphi) = \frac{z^n}{(z\bar{z})^{n/2}} \,, \qquad \bar{\phi} = \phi^* \\ A_z &= -i \frac{n}{2z} \,, \qquad A_{\bar{z}} = A_z^* \end{aligned} \tag{2.4}$$

The second equation is equivalent to

$$A_m \; dx^m = n \; d\varphi \tag{2.5}$$

With the definitions (2.3) the energy functional becomes

$$\mathcal{E} = \frac{1}{2\pi} \int d^2z \left[\frac{1}{2} B^2 + |\mathcal{D}_m \phi|^2 + \frac{\beta}{2} (\phi\bar{\phi} - 1)^2 \right] , \tag{2.6}$$

where $\beta = 2\lambda/e^2$, $\mathcal{D}$ is a covariant derivative with gauge field A and

$$B = \partial_1 A_2 - \partial_2 A_1 = -i(\partial \bar{A} - \bar{\partial} A) = [\mathcal{D}, \bar{\mathcal{D}}] \,. \tag{2.7}$$

It is easy to rewrite this as (see appendix B for the derivation in the noncommutative case)

$$\mathcal{E} = \frac{1}{2\pi} \int d^2z \left[\frac{1}{2} \left(B + (\phi\bar{\phi} - 1) \right)^2 + \mathcal{D}_{\bar{z}} \phi \mathcal{D}_z \bar{\phi} + \frac{\beta - 1}{2} (\phi\bar{\phi} - 1)^2 + \mathcal{T} \right] , \tag{2.8}$$

where

$$\mathcal{T} = \partial_m S^m + B \tag{2.9}$$

$$S^m = \frac{1}{2} \epsilon^{mn} \left(i\phi \mathcal{D}_n \bar{\phi} - i \mathcal{D}_n \phi \bar{\phi} \right) \tag{2.10}$$

Our convention for ϵ^{mn} is that $\epsilon_{12} = 1$.

We will argue below that $\mathcal{T}$ is a topological density, which generalises naturally to the noncommutative case as well. It is easy to see in the commutative case (the same will be true for the noncommutative generalisation) that $\frac{1}{2\pi}\int d^2z\mathcal{T}$ gives the magnetic flux of the vortex and is an integer n. The energy functional for a vortex with winding number n, then, is

$$\mathcal{E} = n + \frac{1}{2\pi}\int d^2z\left[\frac{1}{2}\left(B + (\phi\bar{\phi} - 1)\right)^2 + \mathcal{D}_{\bar{z}}\phi\mathcal{D}_z\bar{\phi} + \frac{\beta - 1}{2}(\phi\bar{\phi} - 1)^2\right]. \tag{2.11}$$

Notice $\mathcal{E}$ is a sum of absolute square terms except the last term. However, when $\beta = 1$, this term drops out and minimum of $\mathcal{E}$ can be obtained if the Bogomolnyi equations are satisfied:

$$\mathcal{D}_{\bar{z}}\phi = 0\,, \qquad \mathcal{D}_z\bar{\phi} = 0\,, \qquad B = 1 - \phi\bar{\phi}\,. \tag{2.12}$$

It is interesting to note that the euclidean action (2.8) can be written in an elegant form using Quillen's superconnection [25] $\mathcal{A}$ defined below

$$E = \int d^2x\left(||\mathcal{F} - 1||^2 + \mathrm{Str}\exp[\mathcal{F}]\right), \tag{2.13}$$

where in the exponential only the term proportional to the volume form contributes, namely

$$\int d^2x\,\mathrm{Str}\exp[\mathcal{F}] = F^+ + \bar{\mathcal{D}}\phi\wedge\mathcal{D}\bar{\phi} - \{F^+, \phi\bar{\phi}\}\,. \tag{2.14}$$

Here $\mathcal{F}$ is the curvature of the superconnection [25] $\mathcal{A}$

$$\mathcal{A} = \begin{pmatrix} d + A^+ & \phi \\ \bar{\phi} & d \end{pmatrix} \tag{2.15}$$

and F^+ is the curvature of A^+. The Bogomolnyi equations (2.12) can be written in a very suggestive form

$$\mathcal{F} - 1 = 0\,. \tag{2.16}$$

3. Noncommutative abelian Higgs model

Now let us consider noncommutative generalisation of these equations. We present our conventions and definitions for noncommutative field theory following [16] (see also appendix A). The energy functional for the noncommutative abelian Higgs model is given by [16]

$$\mathcal{E} = \mathrm{Tr}\left[\frac{1}{2}B^2 + \mathcal{D}_{\bar{z}}\phi\mathcal{D}_z\bar{\phi} + \mathcal{D}_z\phi\mathcal{D}_{\bar{z}}\bar{\phi} + \frac{1}{2}(\phi\bar{\phi} - 1)^2\right], \tag{3.1}$$

where B is now defined as

$$B = [\mathcal{D}, \bar{\mathcal{D}}] = -i\left(\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z\right) - [A_z, A_{\bar{z}}]\,. \tag{3.2}$$

Following the steps described in appendix B, we can recast the energy functional as

$$\mathcal{E} = \mathrm{Tr}\left[\frac{1}{2}(B + (\phi\bar{\phi} - 1))^2 + \mathcal{D}_{\bar{z}}\phi\mathcal{D}_z\bar{\phi} + \mathcal{T})\right], \tag{3.3}$$

where the topological term is now

$$\mathcal{T} = \mathcal{D}_m S^m + B\,, \tag{3.4}$$

with B defined as in (3.2) and

$$\mathcal{D}_m S^m = \partial_m S^m - i[A_m, S^m] \tag{3.5}$$

S^m is defined as before (2.10) with due attention to operator ordering now. The use of the covariant derivative on S^m in (3.4) is necessary since S^m is now gauge-*covariant*.

We will argue in the next section that

$$I = \mathrm{Tr}\,\mathcal{T}\,, \tag{3.6}$$

corresponds to a topological charge. In particular, for the noncommutative Nielsen-Olesen vortex of charge n constructed below, I evaluates to n for any value of the noncommutativity parameter θ.

Bogomolnyi equations. It is clear from (3.3) that the Bogomolnyi equations remain the same as in (2.12), namely

$$\mathcal{D}_{\bar{z}}\phi = 0, \quad \mathcal{D}_z\bar{\phi} = 0, \quad B = 1 - \phi\bar{\phi}\,, \tag{3.7}$$

which are now to be interpreted as operator equations (conventions described in appendix A).

It is difficult to solve the Bogomolnyi equations exactly. We will find the solution in various limits. First we will solve the large θ limit ($\theta \to \infty$ and $1/\sqrt{\theta}$ corrections).

3.1 Large θ

We will now consider the limit when the the noncommutativity parameter θ, defined by

$$[X^1, X^2] = i\theta \tag{3.8}$$

is large. Let us define the following scaled operators

$$X^i = \sqrt{\theta}\widetilde{X^i}\,, \qquad i = 1, 2\,. \tag{3.9}$$

We define the annihilation and creation operators as

$$a = \frac{1}{\sqrt{2}}\left(\widetilde{X^1} + i\widetilde{X^2}\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\widetilde{X^1} - i\widetilde{X^2}\right). \tag{3.10}$$

The scaled complex coordinates $w, \bar{w}$ are defined as

$$z = \sqrt{\theta} w \,, \qquad \bar{z} = \sqrt{\theta} \bar{w} \,. \tag{3.11}$$

In accordance with the fact that the gauge potential A is a 1-form and the magnetic field is a 2-form, the scaled gauge potential $\tilde{A}$ and scaled magnetic field $\tilde{B}$ are given by

$$A = \frac{\widetilde{A}}{\sqrt{\theta}} \,, \qquad B = \frac{\widetilde{B}}{\theta} \,. \tag{3.12}$$

With the above rescalings the energy functional (3.3) becomes

$$\mathcal{E} = \theta \,\mathrm{Tr} \left[\frac{1}{2} \left(\frac{\widetilde{B}}{\theta} + (\phi\bar{\phi} - 1) \right)^2 + \frac{1}{\theta} \mathcal{D}_{\bar{w}} \phi \mathcal{D}_w \bar{\phi} + \frac{\widetilde{B}}{\theta} + \frac{\mathcal{D}_m S^m}{\theta} \right] . \tag{3.13}$$

Here $\partial_w = -\mathrm{Ad} a^\dagger, \partial_{\bar{w}} = \mathrm{Ad} a$ (see appendix A).

Let us now solve the Bogomolnyi equations (3.7) in these rescaled variables order by order in $1/\theta$.

We write the following $1/\theta$-expansion of the Higgs field and the gauge field

$$\begin{aligned} \phi &= \phi_\infty + \frac{1}{\theta} \phi_{-1} + \cdots \\ A &= \frac{1}{\sqrt{\theta}} \widetilde{A} = \frac{1}{\sqrt{\theta}} \left(A_\infty + \frac{1}{\theta} A_{-1} + \cdots \right) . \end{aligned} \tag{3.14}$$

The expansion of $\bar{A}$ is identical to that of A except that $\bar{A}_i$ are hermitean conjugates of A_i. The large-θ expansion of the magnetic field (3.2) is given by

$$B = \frac{1}{\theta} \widetilde{B} = \frac{1}{\theta} B_\infty + \frac{1}{\theta^2} B_{-1} + \cdots \tag{3.15}$$

where

$$\begin{aligned} B_\infty &= -i(\partial_w \bar{A}_\infty - \partial_{\bar{w}} A_\infty) - [A_\infty, \bar{A}_\infty] \\ &= i([a^\dagger, \bar{A}_\infty] + [a, A_\infty]) - [A_\infty, \bar{A}_\infty] \,, \end{aligned} \tag{3.16}$$

$$\begin{aligned} B_{-1} &= -i(\partial \bar{A}_{-1} - \bar{\partial} A_{-1}) - [A_\infty, \bar{A}_{-1}] - [A_{-1}, \bar{A}_\infty] \\ &= i([a^\dagger, \bar{A}_{-1}] + [a, A_{-1}]) - [A_\infty, \bar{A}_{-1}] - [A_{-1}, \bar{A}_\infty] \,. \end{aligned} \tag{3.17}$$

Let us substitute the expansions (3.14), (3.15) in (3.13) or in the Bogomolnyi equations and solve these equations order by order in $1/\theta$.

$o(\theta)$ **Bogomolnyi equations.** The relative orders of the Bogomolnyi equations are easiest to figure out from (3.13). The $o(\theta)$ term in (3.13) gives the leading Bogomolnyi equation give

$$\phi_\infty \bar{\phi}_\infty = 1 \,. \tag{3.18}$$

This equation is solved by Witten [21] and the solution is

$$\phi_\infty = \frac{1}{\sqrt{a^n a^{\dagger n}}} a^n \,, \qquad \bar{\phi}_\infty = a^{\dagger n} \frac{1}{\sqrt{a^n a^{\dagger n}}} \,. \tag{3.19}$$

It is easy to see by simple substitution that this ansatz indeed solves the leading order Bogomolnyi equation (3.18).

It is interesting to note a more general set of solutions of (3.18), namely

$$\begin{aligned} \phi_\infty &= \frac{1}{\sqrt{(a+w_1)\cdots(a+w_n)\,(a^\dagger+\bar{w}_n)\cdots(a^\dagger+\bar{w}_1)}} \left[(a+w_1)\cdots(a+w_n)\right] \\ \bar{\phi}_\infty &= (\phi_\infty)^\dagger \end{aligned} \tag{3.20}$$

It is possible to find a solution for the gauge field and the Higgs field in a $1/\theta$ expansion around this more general solution, although we will not explicitly write it down here. The interpretation of this solution is simple for (a) all w_i coincident and (b) far separated w_i. (a) represents a vortex with all properties the same as (3.19), except that it is translated from the origin of the NC plane to the point w_1. (b) represents n single vortices located at $w_1, w_2, \ldots, w_n$. The kernel of ϕ_∞ is given by a linear span of the n approximately orthonormal coherent states $\{|-w_1\rangle, |-w_1\rangle, \ldots |-w_n\rangle\}$; this corresponds to index $\phi_\infty = n$.

$o(1)$ **Bogomolnyi equations.** At $o(1)$ in (3.13) we get the Bogomolnyi equations involving the covariant derivative of the Higgs field, namely

$$\partial_{\bar{w}}\phi_\infty - i\bar{A}_\infty\phi_\infty = 0 \,, \qquad \partial_w\bar{\phi}_\infty + i\bar{\phi}_\infty A_\infty = 0 \,. \tag{3.21}$$

For the sake of simplicity we will work only with the ϕ equation of motion ($\bar{\phi}$, being hermitean conjugate of ϕ, can be determined from the solution to the ϕ equation of motion).

The ϕ-equation can be written in the operator form as

$$[a, \phi_\infty] = i\bar{A}_\infty\phi_\infty \,. \tag{3.22}$$

Let us recall at this point the action of creation and annihilation operators of the harmonic oscillator on the one particle Hilbert space.

$$a|m\rangle = \sqrt{m}|m-1\rangle \,, \qquad a^\dagger|m\rangle = \sqrt{m+1}|m+1\rangle \,. \tag{3.23}$$

For future purposes, it is useful to write down the action of the Higgs field on the one particle Hilbert space. The Higgs field configuration for the vortex solution is written in terms of harmonic oscillator creation and annihilation operators. Using (3.23), it is easy to see that the action of the Higgs field on the Hilbert space is

$$\phi|m\rangle = |m-n\rangle \,. \tag{3.24}$$

Now let us look at the subleading equation, that is (3.22). Using the above results it is easy to derive the gauge field $\bar{A}_\infty$ (A_∞ is given by its hermitean conjugate). Thus

$$\begin{aligned}\bar{A}_\infty &= -i\frac{1}{\sqrt{N+1}}a\left(\sqrt{N}-\sqrt{N+n}\right)\\ A_\infty &= i\left(\sqrt{N}-\sqrt{N+n}\right)a^\dagger\frac{1}{\sqrt{N+1}}\,,\end{aligned} \tag{3.25}$$

where $N = a^\dagger a$ is the number operator.

To gain a better understanding of the operator solutions (3.19) and (3.25) let us evaluate their expectation values in a coherent state

$$|w\rangle = e^{wa^\dagger}|0\rangle\,. \tag{3.26}$$

By using the result

$$\langle w|f(a,a^\dagger)|w\rangle = \langle 0|f(a+w,a^\dagger+\bar{w})|0\rangle \tag{3.27}$$

it is easy to see that in the large w limit or equivalently in the large $\langle N\rangle$ limit, the expectation values become

$$\begin{aligned}\langle w|\phi_\infty|w\rangle &= \exp(in\varphi),\\ \langle w|\bar{A}_\infty|w\rangle &= i\frac{n}{2\bar{w}}\,, \qquad \langle w|A_\infty|w\rangle = -i\frac{n}{2w}\,.\end{aligned} \tag{3.28}$$

The large distance behaviour here is exactly the same as the large distance behaviour of the usual Nielsen-Olesen vortex (2.4).[1]

In a way, this result is expected because in the large-θ limit we have ignored the derivative terms and then in the large-$\langle N\rangle$ limit the asymptotic behaviour of the vortex solution becomes exact. Behaviour of the vortex solution in the finite domain of the w plane, that is, for finite values of $\langle N\rangle$ depends on the competition between the kinetic energy terms and the potential energy terms in the energy functional. Exact solution to the equations in the large-θ expansion, which essentially ignores the kinetic energy effect, of the Bogomolnyi equations, reduces to the potential energy minimisation in the large-$\langle N\rangle$ limit. We will show below that (3.6) evaluates to n for this solution. Therefore, this solution carries topological charge, which is determined by the quantized magnetic flux through the vortex solution.

The magnetic field. From (3.16) we see that B_∞ is given entirely in terms of A_∞. Using the solution (3.25) in (3.16) (details in appendix C) we get

$$B_\infty = n|0\rangle\langle 0|\,, \tag{3.29}$$

[1] In stead of calculating expectation values in coherent states one could alternatively evaluate the Weyl-Moyal (inverse) map of the operator solutions to find the classical functions on the NC plane; the large distance behaviour obtained this way is the same as that in (3.28) or (2.4).

JHEP09(2000)018

where $|0\rangle\langle 0|$ is a projection operator onto the vacuum state. It is curious that the leading term in the large-θ expansion of the magnetic field has such a remarkably simple form. It is also interesting to note that the trace of B_∞ in the one particle Hilbert space gives us exactly the integer n as desired. We will elaborate more on this when we will discuss the topology of our solution. In terms of the original unscaled coordinates this vacuum projection operator is essentially a δ-function. However, in the scaled variables the vacuum projection operator is represented by the ground state wavefunction, i.e., by the gaussian. Significance of this will be discussed in the next section.

$o(1/\theta)$ and higher orders. So far we have looked at the leading and the first subleading term in the large-θ expansion of the Bogomolnyi equations. Here we will look at the higher corrections to the Higgs field ϕ as well as the gauge field A. It is interesting to see that the large-θ correction are quite small and the convergence of the solution is remarkably fast.

The $o(1/\theta)$ equation of motion is given by

$$B_\infty = -\phi_\infty \bar{\phi}_{-1} - \phi_{-1}\bar{\phi}_\infty \,. \tag{3.30}$$

Notice that since we have already determined B_∞, we can use the above equation to solve for ϕ_{-1}. We get

$$\phi_{-1} = -\frac{n}{2}|0\rangle\langle n|\,, \qquad \bar{\phi}_{-1} = -\frac{n}{2}|n\rangle\langle 0|\,. \tag{3.31}$$

As mentioned in the beginning of this subsection the solution has a very good convergence property. To see this it is instructive to write the leading order solution (3.19) as follows:

$$\phi_\infty = \sum_{m=0}^{\infty} |m\rangle\langle m+n|\,, \qquad \text{and} \qquad \bar{\phi}_\infty = \sum_{m=0}^{\infty} |m+n\rangle\langle m|\,. \tag{3.32}$$

Thus the leading order solution involves an infinite sum whereas the first correction contains only one term as can be seen from (3.31).

The subleading correction to the gauge field solution is obtained by solving the $o(\theta^{-2})$ equation

$$\bar{\partial}\phi_{-1} - i\bar{A}_\infty\phi_{-1} - i\bar{A}_{-1}\phi_\infty = 0 \tag{3.33}$$

The solution is

$$\bar{A}_{-1} = -i\frac{n}{2}\sqrt{n+1}|0\rangle\langle 1|\,. \tag{3.34}$$

Substituting this result in (3.17) we can determine first subleading correction to the magnetic field

$$B_{-1} = n(n+1)\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right). \tag{3.35}$$

Note that the correction has a vanishing trace, ensuring that

$$\mathrm{Tr}\left(B_\infty + \frac{1}{\theta}B_{-1}\right) = \mathrm{Tr}\,B_\infty = n\,. \tag{3.36}$$

3.2 Finite θ

Having constructed the vortex solution for large θ, we now ask what happens to this solution at finite θ. To do this let us look at the Seiberg-Witten(SW) map for the noncommutative abelian Higgs system.

Recall the SW map for the pure Yang-Mills theory is

$$\begin{aligned}\hat{S}(A) &\equiv \hat{A}_i(A)A_i - \frac{1}{4}\theta^{kl}\{A_k, \partial_l A_i + F_{li}\} + \mathcal{O}(\theta^2)\\ \hat{S}(\lambda) &\equiv \hat{\lambda}(\lambda, A)\lambda + \frac{1}{4}\theta^{kl}\{\partial_k\lambda, A_l\} + \mathcal{O}(\theta^2)\,.\end{aligned} \tag{3.37}$$

We wish to carry out this exercise for ϕ, i.e., we look for a map $\hat{S}$

$$(\phi, A) \mapsto \hat{S}(\phi, A) \equiv (\hat{\phi}, \hat{A}) \tag{3.38}$$

such that

$$\hat{S}(\phi + i\lambda\phi, A + d\lambda) = \left(\hat{\phi} + i\hat{\lambda}\phi, \hat{A} + d\hat{\lambda} + i\left(\hat{\lambda}\hat{A} - \hat{A}\hat{\lambda}\right)\right), \tag{3.39}$$

where both the map $\hat{S}$ and the noncommutative gauge transformation parameter $\hat{\lambda}$ are to be found so as to satisfy the above equation. It turns out that the map for $\hat{A}$ and $\hat{\lambda}$ given in (3.37) is unaltered. The map for ϕ is

$$\hat{\phi} = \phi - \frac{1}{2}\theta\epsilon^{kl}A_k\partial_l\phi + o(\theta^2)\,. \tag{3.40}$$

The SW map can be used for determining the change in the fields at any value $\theta = \theta_0$ due to small increment $\delta\theta$. At $\theta = 0$, right hand side of (3.37) and (3.40) contain ordinary products of the fields, but if (3.37) and (3.40) are used for determining a small increment at $\theta = \theta_0$ then right hand side of these equations contain $\star$-products.

There is an important difference between the SW map for the Higgs field ϕ and that for the gauge field A. Whereas the SW map for the gauge field A is nonlinear in A that for the Higgs field ϕ is linear in ϕ. It is the nonlinearity of the SW map for the gauge field which was useful [4] in relating the noncommutative gauge theory action to the Born-Infeld action for the ordinary gauge theory. It is easy to see by successive transformations that the SW map for ϕ remains linear in ϕ but becomes a nonlinear function of A. Therefore, it is tempting to conjecture that the Higgs action in terms of the ordinary Higgs field would still be of the same form, albeit multiplied by complicated functions of the gauge field A.

By using the above result, it is easy to see that the equations of motion, written in terms of the ordinary commutative fields, remain the same at $r \to \infty$ as in the usual abelian Higgs model case. Therefore, the asymptotic form of the vortex solution, written above, is valid. Though the solution in the noncommutative problem will differ from the abelian case in the bulk, for our purposes here, especially in the next section, where we discuss the topology of the vortex solution, detailed form of the solution in the bulk is not relevant.[2]

4. Topology

It is interesting to ask what happens to topology when one studies noncommutative gauge theories coupled to matter. Below we show that the topological charge I in (3.6) is independent of θ and therefore the configuration space of noncommutative theory splits into the same topological sectors as in the commutative case. We first show this for small θ.

4.1 Small-θ expansion

We rewrite (3.6) in the Moyal form (by that we mean using ordinary functions and star products). Thus,

$$I = \frac{1}{2\pi} \int d^2 z (\mathcal{D}_m S^m + B) \,, \tag{4.1}$$

where the covariant derivatives and the magnetic field B are defined, in the Moyal formalism, in (A.8) in appendix A.

Besides the explicit θ-dependence involved in the star product, S^m and B also have an expansion in terms of θ since they are built out of ϕ and A_i which are solutions of the Bogomolnyi equations (we imagine writing these equations here in the Moyal form, therefore explicitly containing θ, and solving them iteratively in small θ). We write

$$\begin{aligned} \phi &= \phi_0 + \theta \phi_1 + \cdots \\ A_i &= A_{i,0} + \theta A_{i,1} + \cdots \end{aligned} \tag{4.2}$$

The small-θ expansion for $B = \epsilon^{kl}(\partial_k A_l - i A_k \star A_l)$ and the topological charge (4.1) are

$$\begin{aligned} B &= B_0 + \theta B_1 + \cdots \\ I &= I_0 + \theta I_1 + \cdots \end{aligned} \tag{4.3}$$

The zero-order term evaluates to

$$I_0 = \frac{1}{2\pi} \int d^2 x B_0 = \frac{1}{2\pi} \oint A_{i,0} dx^i = n \tag{4.4}$$

[2] A numerical vortex solution has been constructed in [26] in the context of a somewhat different action.

This follows from (2.4) since in the zero-th order in θ the Bogomolnyi equations are identical to those of the ordinary abelian Higgs model. We have also used the fact that at zero-th order

$$\int d^2x \ [\mathcal{D}_m S^m]_0 = \oint r d\varphi S_{r,0}|_{r=\infty} = 0 \tag{4.5}$$

as can be seen explicitly from the asymptotic form (2.4).

We now carry out the iterative solution to first nontrivial order in θ. Thus, for example,

$$B_1 = \frac{1}{2}\epsilon^{kl}\epsilon^{ij}\partial_k A_{i,0}\partial_l A_{j,0} + \epsilon^{kl}\partial_k A_{l,1} = -\left(\frac{i}{2}\epsilon^{kl}\partial_k\phi_0\partial_l\bar{\phi}_0 + \phi_0\bar{\phi}_1 + \phi_1\bar{\phi}_0\right). \tag{4.6}$$

It is easy to see that the contribution of the magnetic field to I_1 becomes a total derivative. The expression after the first equality is given by

$$\begin{aligned} B_1 &= \partial_k f^k \\ f^k &\equiv \epsilon^{kl}\left(A_{l,1} + \frac{1}{2}\epsilon^{ij}A_{i,0}\partial_l A_{j,0}\right). \end{aligned} \tag{4.7}$$

The contribution of B_1 to the topological charge at this order, therefore, is

$$I_1 = \int d^2x \ B_1 = \oint r \ d\varphi \ f_r|_{r=\infty} = 0\,. \tag{4.8}$$

It is easy to see that f_r vanishes as $f_r \sim 1/r^3$. Similar arguments can be made to show that the contribution of $\mathcal{D}_m S^m$ becomes a total derivative too, and the surface term at $r = \infty$ vanishes.

These arguments, namely that (a) the integrals involved in I_n all become surface terms, and (b) the surface terms vanish at $r = \infty$, in fact generalize to all higher orders in θ. The proof of (a) is s straight-forward generalisation of the first-order calculation; regarding (b) we need to only use the fact, proved at the end of the previous section that higher $\phi_n, A_{i,n}$ have a faster fall-off at $r = \infty$ than the zero order solutions. Regarding the rate of fall-off of the surface terms in successive I_n it is easy to see from a scaling argument (using scaled coordinates $(1/\sqrt{\theta})x^i$) that order θ^n terms are down by $2n$ powers of derivative, or equivalently by $1/r^{2n}$ in the large-r limit.

To summarise, we see that the higher order terms in the small-θ expansion of I all vanish and therefore do not modify the topological charge.

4.2 Large-θ expansion

We now show that the topological charge (3.6) is the integer n also in the large-θ limit and would like to argue that the successive orders in the $1/\theta$ expansion do not modify this result.

From the expression (3.29), namely

$$B_\infty = n|0\rangle\langle 0| \,, \tag{4.9}$$

it is easy to see that

$$\mathrm{Tr}\, B_\infty = n \,. \tag{4.10}$$

We emphasize here that to see the result (4.10), one must consider the subleading $1/\theta$ terms as in the last section. In the $\theta = \infty$ limit, as indicated in (3.15), $B = 0$. In other words, since "Tr" in (4.10) is actually $\theta \int d^2 z$ in the original coordinates, it is essential to keep the $1/\theta$ terms in B.

By using calculations similar to appendix C, it is easy to see that

$$\mathrm{Tr}\, \mathcal{D}_m S^m = 0 \tag{4.11}$$

using the leading order expressions in large θ.

Thus, if one writes

$$I = I_\infty + \theta^{-1} I_{-1} + \cdots \tag{4.12}$$

we have

$$I_\infty = n \,. \tag{4.13}$$

It is easy to check, using results from the last section, e.g. (3.35), that

$$I_{-1} = 0 \,. \tag{4.14}$$

Although we have not checked it in explicit detail, it is easy to argue, in keeping with our result for small θ, that the higher order terms in the large-θ expansion will all vanish too. This implies

$$I = n \tag{4.15}$$

nonperturbatively.

4.3 Comments

We would like to make several comments:

1. Although we have ostensibly shown that I evaluates to an integer by using the on-shell vortex solution, it is clear from our small-θ arguments that the expression I will evaluate to an integer for all *off-shell* configurations which satisfy the conditions (2.4) at $r = \infty$.

2. The topological charge I we have introduced can be identified with RR charge in the context of vortex solutions on brane-antibrane systems. This fact has been shown by vertex operator calculations in [27]. This provides additional evidence the quantity I we have introduced must be quantized.

JHEP09(2000)018

3. It has been remarked in [21] that the topological charge of the vortex is actually the same as the index of ϕ:

$$I = \iota(\phi) = \text{dim ker } \phi - \text{dim coker } \phi\,. \tag{4.16}$$

Note that such a relation automatically implies a "quantization" (integer-valued-ness) of the topological charge I. It is important to appreciate that the "quantization" here does not refer to the usual quantization (specified by a finite $\hbar$) but rather to noncommutative field theory. In a commutative field theory, with or without $\hbar$, there is no notion of a kernel of ϕ. It is a characteristic feature of NCFT's that one can have "quantization" conditions in a classical field theory.

4. One can see in a "semiclassical" sense how I is also related to the Atiyah-Singer index of the Dirac operator in the gauge field background of the vortex. We skip the details here, but the main point is that if one considers the normal mode of a fermion field $\xi_m = z^m$ and adiabatically turns on the background gauge field representing our vortex solution, then the normal mode gets transformed to

$$z^m \mapsto \ \exp i \int A \cdot z^m = z^{m-n}\,. \tag{4.17}$$

By the standard arguments relating to spectral flow and the index, one can see that the index of the Dirac operator is n since the "Fermi sea" shifts by n.

5. It would be very interesting to figure out why our topological charge I is equal (4.16) to the index of ϕ. It is natural, e.g. from the viewpoint of the superconnection [25], that the Dirac equation for the fermion should be considered in the background of both the gauge field and the Higgs field. In other words, the fermion zero modes should satisfy, schematically

$$\not{\mathcal{D}}\xi \equiv (\not{\partial} + \not{A} + \phi)\xi = 0\,. \tag{4.18}$$

It is possible that the toplogical charge I actually measures the index of $\not{\mathcal{D}}$. In that case, in the large-θ limit, $\not{\partial} + \not{A}$ will pick up a factor of $1/\sqrt{\theta}$ and decouple, leaving just the index of ϕ.[3]

6. The zero-th order equation for ϕ in the large-θ limit, namely (3.18), is the equation of a *fuzzy circle* in the configuration space. The calculation in appendix C can be interpreted to mean, on the other hand, that the topological charge I receives contribution from states with expectation values

$$\langle a^\dagger a\rangle = M \Rightarrow \langle (X^1)^2 + (X^2)^2\rangle = 2\theta\left(M + \frac{1}{2}\right), \tag{4.19}$$

[3]We thank Mike Douglas for a discussion on this point.

where $M \to \infty$. Equation (4.19) can be regarded as a fuzzy circle in "coordinate space".[4] Thus, I appears to characterise maps from a fuzzy circle to a fuzzy circle.

We would also like to make a few remarks about the qualitative nature of the vortex solution we have found. Let us, in particular, discuss how some features of the magnetic field (3.29), (3.35) in the large-θ expansion could be arrived at by the following physical reasoning.

It can be seen from the form of A_∞ and $\bar{A}_\infty$ in (3.25) and the expression of B_∞ in (3.16) that magnetic field has to involve equal number of creation and annihilation operators.

Now, for the n vortex solution, ϕ annihilates the ground state as well as $n-1$ excited states in the one particle Hilbert space. Topology of the solution is obtained by taking trace of B_∞ over the whole Hilbert space. This can also be done as proposed by Witten [21] by determining the index of ϕ. The index of ϕ is n since the lowest n states in the Hilbert space are in the kernel of ϕ. So we would expect that B_∞ can be written in terms of a linear combination of the projection operator $|i\rangle\langle i|$, where $i = 0, \ldots, n-1$.

As is well known (see, e.g. [15, 17]), these projection operators are represented in the Moyal form (in the sense of appendix A) in terms of Laguerre functions, related to harmonic oscillator wavefunctions. In particular, $|i\rangle\langle i|$ is represented by the i-th excited state wavefunction. All these wavefunctions have nodes except the ground state wavefunction. If the magnetic field is written in terms of a particular $|i\rangle\langle i|$ for some $i \neq 0$ then the magnetic field develops a zero in a finite region in the w plane and as a consequence of the Bogomolnyi equations, the Higgs field relaxes to its vacuum value. This configuration is allowed only at asymptotic infinity in the w plane. One can, therefore, rule out the possibility that the magnetic field has a form $|i\rangle\langle i|$ for some $i \neq 0$ since it contradicts with the minimal energy ansatz of the Bogomolnyi limit. One may suspect that this may correspond to the n vortex solution with n single vortex solutions centred at different locations in w plane. In fact, this is also not possible as the i-th excited state wavefunction changes sign after encountering a node, whereas change in the sign of the magnetic field would correspond to an anti-vortex configuration. It is well known that a vortex-antivortex configuration does not satisfy Bogomolnyi condition. This still leaves the possibility, however, that the magnetic field is a linear combination of $|i\rangle\langle i|$ for i belonging to a certain subset of the Hilbert space in such a way that the resulting functional form of B never develops a zero. Profile of the magnetic field of the vortex that we seem to get at the leading order indicates that we have a single vortex of vorticity n sitting at the origin. A careful look at the subleading corrections to B in the large-θ limit,

[4] It is interesting to note the appearance of the zero point energy in the expression for radius of the fuzzy circle.

however, shows that the general profile of the magnetic field does allow for the linear combinations of $|i\rangle\langle i|$ for $i \neq 0$. As has been shown at the end of previous subsection these corrections to the magnetic field do not affect the value of the topological charge.

5. Higgs coupling to diagonal U(1)

So far we have considered the NC Higgs model which has a nontrivial action only of the left U(1). Here we will consider a model which has left-right action. In other words, in NC abelian Higgs model, the Higgs field couples to left and right abelian gauge fields which are different fields in general, i.e., NC abelian Higgs model has $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ gauge symmetry. In case of left module, $\mathrm{U}(1)_R$ gauge field is set to zero. In this section we will consider a model where left and right U(1) gauge fields are identified with each other. This is of interest because this model does not have a commutative analogue. For another recent discussion of this model, see [28].

It is easy to generalize (2.12) to this case and they are given by

$$D_{\bar{z}}\phi = 0\,, \qquad D_z\bar{\phi} = 0\,, \qquad B = 1 - \phi\bar{\phi} + \bar{\phi}\phi\,, \tag{5.1}$$

where the covariant derivative is defined as

$$D_{\bar{z}}\phi = \partial_{\bar{z}}\phi - iA_{\bar{z}}\phi + i\phi A_{\bar{z}} \tag{5.2}$$

and similarly for $\bar{\phi}$. These equations can be rewritten in terms of the one form gauge field $\mathcal{A}$ as

$$[\mathcal{A}, \bar{\mathcal{A}}] = \phi\bar{\phi} + \bar{\phi}\phi\,, \tag{5.3}$$

where, $[\mathcal{A}, \bar{\mathcal{A}}] = 1 - B$ [15].

This equation supports exact solution unlike the left module NC abelian Higgs model. To arrive at this solution let us notice the following fact about the commutator $[\chi^2, \psi]$, where χ and ψ are any two operators.

$$[\chi^2, \psi] = \chi[\chi, \psi] + [\chi, \psi]\chi = \{\chi, [\chi, \psi]\}\,. \tag{5.4}$$

This relation can be used to convert the anticommutator on the right hand side of the Bogomolnyi equation into a commutator. For our purpose, we choose $\chi = \xi^2$, $\psi = \bar{\xi}^2$, and $\phi = 2\xi$ and $\bar{\phi} = [\xi, \bar{\xi}^2]$. Substituting this in the Bogomolnyi equations, it is easy to see that these equations are solved by

$$\mathcal{A} = \sqrt{2}\xi^2, \qquad \bar{\mathcal{A}} = \sqrt{2}\bar{\xi}^2\,. \tag{5.5}$$

Putting this into the second Bogomolnyi equation gives us the following relations

$$[\xi^2, \xi] = 0, \qquad [\bar{\xi}^2, [\xi, \bar{\xi}^2]] = 0\,. \tag{5.6}$$

Of these equations, first one is trivially satisfied whereas the second equation is also satisfies if

$$[\xi, \bar{\xi}^2] = f(\bar{\xi})\,. \tag{5.7}$$

In particular, for the choice of $f(\bar{\xi}) = 2\bar{\xi}$ the equation reduces to

$$[\xi, \bar{\xi}] = 1\,. \tag{5.8}$$

Thus the Bogomolnyi equations are solved by

$$\mathcal{A} = \sqrt{2}\xi^2\,, \qquad \bar{\mathcal{A}} = \sqrt{2}\bar{\xi}^2\,, \qquad \phi = 2\xi\,, \qquad \bar{\phi} = 2\bar{\xi}\,. \tag{5.9}$$

An example of solution to (5.8) is

$$\xi = a\,, \qquad \bar{\xi} = a^\dagger\,. \tag{5.10}$$

Such a solution is obviously not normalizable. However using arguments similar to that in [21] we see that this solution has nontrivial winding number (=1), because, firstly (1) $\xi \sim r\exp(i\varphi)$ in the semiclassical limit and secondly, (2) dim ker $\xi-$ dim ker $\bar{\xi} = 1 - 0 = 1$. It would be interesting to get this winding number from a suitably regularised topological charge.

Acknowledgments

One of us (S.R.W.) would like to thank Luis Alvarez-Gaume and Avinash Dhar for discussions in the initial stages of this work. D.P.J. would like to thank Amol Dighe for discussion and Theory Division of CERN for hospitality. G.M. would like to thank Mike Douglas for a discussion regarding the topological index.

A. Noncommutative field theory: operator conventions

We collect below the conventions and some results that are needed for the purposes of our paper. For more details, see [16].

The Weyl-Moyal map. We consider a one-particle Hilbert space $\mathcal{H}$ carrying a representation of the Heisenberg algebra

$$[X^1, X^2] = i\,. \tag{A.1}$$

For f a function on R^2

$$f(x^1, x^2) = \frac{1}{2\pi}\int dk_1\; dk_2\; \tilde{f}(k_1, k_2)\exp[ik_1x^1 + ik_2x^2] \tag{A.2}$$

we define an operator $\hat{M}(f)$ on $\mathcal{H}$

$$f \mapsto \hat{M}(f) \tag{A.3}$$

by the rule

$$\hat{M}(f) = \frac{1}{2\pi}\int dk_1\ dk_2\ \tilde{f}(k_1, k_2)\exp[ik_1X^1 + ik_2X^2]\,, \tag{A.4}$$

where X^1, X^2 are the operators in (A.1). It is easy to see that

$$\begin{aligned} \operatorname{Tr}\hat{M}(f) &= \frac{1}{2\pi}\int d^2x f(x^1, x^2) \\ \hat{M}(f)\hat{M}(g) &= \hat{M}(f\star g) \\ \hat{M}\left(\frac{\partial f}{\partial x^i}\right) & (i\theta)^{-1}\epsilon_{ik}[X^k, \hat{M}(f)]\,, \end{aligned} \tag{A.5}$$

where

$$f\star g(x^1, x^2) = \left(\exp\left[\frac{i}{2}\theta\epsilon^{kl}\frac{\partial}{\partial x^k}\frac{\partial}{\partial y^l}\right] f(x^1, x^2)g(y^1, y^2)\right)\Big|_{y=x}\,. \tag{A.6}$$

Clearly, NCFT can be defined either in terms of operators on $\mathcal{H}$ or on ordinary functions whose multiplication is defined in the sense of star product. We use the operator approach in most of our paper. Thus, for example, the equation (3.1) is written in the operator language; its alternative form (which we will call the Moyal form) will be

$$\mathcal{E} = \frac{1}{2\pi}\int d^2z\left[\frac{1}{2}B\star B + \mathcal{D}_{\bar{z}}\phi\star\mathcal{D}_z\bar{\phi} + \mathcal{D}_z\phi\star\mathcal{D}_{\bar{z}}\bar{\phi} + \frac{1}{2}(\phi\star\bar{\phi} - 1)\star(\phi\star\bar{\phi} - 1)\right]. \tag{A.7}$$

where

$$\begin{aligned} D_i\phi &= \partial_i\phi - iA_i\star\phi \\ B &= \partial\bar{A} - \bar{\partial}A - i(A\star\bar{A} - \bar{A}\star A) \end{aligned} \tag{A.8}$$

Conventions for the noncommutative $\mathrm{U}(1)\times\mathrm{U}(1)$ **gauge theory.** Note that for NC gauge theory, $g\phi \neq \phi g$ even when g is (an operator representative of) a U(1) transformation. Thus it is important to distinguish between the left U(1) from the right U(1).

We define the left- and right-gauge fields as A_i and A'_i and the corresponding gauge transformations as δ_λ and $\delta_{\lambda'}$ respectively. Our conventions for the left U(1) are

$$\begin{aligned} \delta_\lambda A_i &= \partial_i\lambda + i[\lambda, A_i] \\ \delta_\lambda\phi &= i\lambda\phi \\ \delta_\lambda\bar{\phi} &= -i\bar{\phi}\lambda\,. \end{aligned} \tag{A.9}$$

Similarly for the right U(1)

$$\begin{aligned}
\delta_{\lambda'} A'_i &= \partial_i \lambda' + i[\lambda', A'_i] \\
\delta_{\lambda'} \phi &= -i\phi\lambda' \\
\delta_{\lambda'} \bar{\phi} &= i\lambda'\bar{\phi}\,.
\end{aligned} \tag{A.10}$$

The covariant derivatives, when both the gauge fields are non-zero, are given by

$$\begin{aligned}
D_i\phi &= \partial_i\phi - iA_i\phi + i\phi A'_i \\
D_i\bar{\phi} &= \partial_i\bar{\phi} + i\bar{\phi}A_i - iA'_i\bar{\phi}\,.
\end{aligned} \tag{A.11}$$

In the above formulae we mean the derivatives and products as in the operator formulation explained above.

In sections 2, 3 and 4, we consider only the left U(1), by putting $A'_i = 0 = \lambda'$. This has a commutative limit (the same as the right U(1)). In section 5, we put $A_i = A'_i$ (the gauge transformations $\lambda = \kappa = \lambda'$) which corresponds to the diagonal U(1). Clearly the diagonal U(1) does not have a commutative counterpart since the relevant gauge transformation, for example of ϕ, is

$$\delta_\kappa\phi = [\delta_\lambda\phi + \delta_{\lambda'}\phi]_{\lambda=\lambda'=\kappa} = i[\kappa, \phi] \tag{A.12}$$

which disappears in the commutative limit.

B. Derivation of the topological density

In order to derive (3.3) we need to show that

$$(D_m S^m + B) + 2\bar{D}\phi D\bar{\phi} + \frac{1}{2}(B + (\phi\bar{\phi} - 1))^2 = D\phi\bar{D}\bar{\phi} + \bar{D}\phi D\bar{\phi} + \frac{1}{2}B^2 + \frac{1}{2}(\phi\bar{\phi} - 1)^2\,. \tag{B.1}$$

This is equivalent to

$$D_m S^m = D\phi\bar{D}\bar{\phi} - \bar{D}\phi D\bar{\phi} - \frac{1}{2}(\phi\bar{\phi}B + B\phi\bar{\phi})\,. \tag{B.2}$$

We start by observing that

$$\begin{aligned}
D_z S^z &= \frac{1}{2}\left(\partial(\phi\bar{D}\bar{\phi} - \bar{D}\phi\bar{\phi}) - i[A, \phi\bar{D}\bar{\phi} - \bar{D}\phi\bar{\phi}]\right) \\
&= \frac{1}{2}\left(D\phi\bar{D}\bar{\phi} + \phi D(\bar{D}\bar{\phi}) - D(\bar{D}\bar{\phi})\bar{\phi} - \bar{D}\phi D\bar{\phi}\right).
\end{aligned} \tag{B.3}$$

Similarly

$$D_{\bar{z}} S^{\bar{z}} = -\frac{1}{2}\left(\bar{D}\phi D\bar{\phi} + \phi\bar{D}(D\bar{\phi}) - \bar{D}(D\bar{\phi})\bar{\phi} - D\phi\bar{D}\bar{\phi}\right) \tag{B.4}$$

Using these, and the fact that $[D, \bar{D}]\phi = B\phi, [D, \bar{D}]\bar{\phi} = -\bar{\phi}B$ we get (B.2).

C. Calculation of B_∞

We give some steps in the evaluation of (3.16).

We give details how to calculate traces of quantities like

$$t \equiv [a, C] \tag{C.1}$$

for some operator C. Here a is the usual annihilation operator, satisfying

$$a|m\rangle = \sqrt{m}|m-1\rangle \quad \forall m = 1, \dots \tag{C.2}$$

Formally,

$$\begin{aligned}\mathrm{Tr}\, t &= \sum_{m=0}^{M} \langle m| a\, C - C\, a|m\rangle \\ &= \sum_{m=0}^{M} \sqrt{m+1}\langle m+1|C|m\rangle - \sum_{m=1}^{M} \sqrt{m}\langle m|C|m-1\rangle \end{aligned}\tag{C.3}$$

where eventually we should take the limit $M \to \infty$ (this regulator has been used in [17]). There are two ways in which we can proceed from here:

Method 1: shifting the summation variable in the second term, we get

$$\mathrm{Tr}\, t = \sqrt{M+1}\langle M+1|C|M\rangle \tag{C.4}$$

The calculation from this viewpoint reduces to the fuzzy circle at the boundary $\langle a^\dagger a\rangle = M \to \infty$. This actually provides a noncommutative version of Stokes' theorem (recall that Ad a actually plays the role of a derivative operator).

Method 2: in this method, we note that the first summation has a contribution from the ground state which is missing from the second summation. We take care of this fact by writing

$$C\ a|m\rangle = \sqrt{m} C(1 - |0\rangle\langle 0|)|m-1\rangle \tag{C.5}$$

Of course, the answers in both the methods are the same.

The first two terms in (3.16) are already of the form (C.1) (or hermitean conjugate). The two methods described above also work for the third term in a similar fashion.

According to method 1, the trace of (3.16) gets evaluated as

$$\mathrm{Tr}\, B_\infty = \lim_{M=\infty} 2\sqrt{M}(\sqrt{M+n} - \sqrt{M}) = n\,. \tag{C.6}$$

The contributions come entirely from the first two terms in (3.16).

According to method 2, the first and second terms in (3.16) give

$$2\left[(N - \sqrt{N(N+n)})(1 - |0\rangle\langle 0|) - (N+1-\sqrt{(N+1)(N+n+1)})\right]\,, \tag{C.7}$$

whereas the third term gives

$$-(\sqrt{N}-\sqrt{N+n})^2(1-|0\rangle\langle 0|)+(\sqrt{N+1}-\sqrt{N+n+1})^2\,. \tag{C.8}$$

Combining, we get

$$B_\infty = n|0\rangle\langle 0|\,, \tag{C.9}$$

namely equation (3.29).

References

[1] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[2] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[3] M.R. Douglas, *Two lectures on D-geometry and noncommutative geometry*, hep-th/9901146.

[4] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[5] D. Bigatti and L. Susskind, *Magnetic fields, branes and noncommutative geometry*, *Phys. Rev.* **D 62** (2000) 066004 [hep-th/9908056].

[6] F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, *Mixed branes and M(atrix) theory on noncommutative torus*, hep-th/9803067; *Noncommutative geometry from strings and branes*, *J. High Energy Phys.* **02** (1999) 016 [hep-th/9810072].

[7] Y.-K.E. Cheung and M. Krogh, *Noncommutative geometry from 0-branes in a background B-field*, *Nucl. Phys.* **B 528** (1998) 185 [hep-th/9803031].

[8] V. Schomerus, *D-branes and deformation quantization*, *J. High Energy Phys.* **06** (1999) 030 [hep-th/9903205].

[9] A.H. Chamseddine and J. Frohlich, *Some elements of Connes' noncommutative geometry and space-time geometry*, hep-th/9307012.

[10] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Non-commutative world-volume geometries: branes on* SU(2) *and fuzzy spheres*, *J. High Energy Phys.* **09** (1999) 023 [hep-th/9908040].

[11] N. Nekrasov and A. Schwarz, *Instantons on noncommutative R^4 and $(2,0)$-superconformal six dimensional theory*, *Comm. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[12] A. Dhar, G. Mandal and S.R. Wadia, *Nonrelativistic fermions, coadjoint orbits of W_∞ and string field theory at $c=1$*, *Mod. Phys. Lett.* **A 7** (1992) 3129 [hep-th/9207011]; *W_∞ coherent states and path integral derivation of bosonization of nonrelativistic*

fermions in one-dimension, *Mod. Phys. Lett.* **A 8** (1993) 3557 [hep-th/9309028]; *A time dependent classical solution of $c = 1$ string field theory and nonperturbative effects*, *Int. J. Mod. Phys.* **A 8** (1993) 3811 [hep-th/9212027].

[13] A. Dhar, G. Mandal and S.R. Wadia, *String field theory of two-dimensional QCD: a realization of W_∞ algebra*, *Phys. Lett.* **B 329** (1994) 15 [hep-th/9403050].

[14] S. Minwalla, M.V. Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, hep-th/9912072.

[15] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, *J. High Energy Phys.* **05** (2000) 020 [hep-th/0003160].

[16] L. Alvarez-Gaume and S.R. Wadia, *Gauge theory on a quantum phase space*, hep-th/0006219.

[17] D.J. Gross and N.A. Nekrasov, *Monopoles and strings in noncommutative gauge theory*, *J. High Energy Phys.* **07** (2000) 034 [hep-th/0005204].

[18] K. Dasgupta, S. Mukhi and G. Rajesh, *Noncommutative tachyons*, *J. High Energy Phys.* **06** (2000) 022 [hep-th/0005006].

[19] J.A. Harvey, P. Kraus, F. Larsen and E.J. Martinec, *D-branes and strings as noncommutative solitons*, *J. High Energy Phys.* **07** (2000) 042 [hep-th/0005031].

[20] A. Sen, *Stable non-BPS bound states of BPS D-branes*, *J. High Energy Phys.* **08** (1998) 010 [hep-th/9805019]; *Tachyon condensation on the brane antibrane system*, *J. High Energy Phys.* **08** (1998) 012 [hep-th/9805170]; *Descent relations among bosonic D-branes*, *Int. J. Mod. Phys.* **A 14** (1999) 4061 [hep-th/9902105].

[21] E. Witten, *Noncommutative tachyons and string field theory*, hep-th/0006071.

[22] A. Sen, SO(32) *spinors of type-I and other solitons on brane-antibrane pair*, *J. High Energy Phys.* **09** (1998) 023 [hep-th/9808141]; *Non-BPS states and branes in string theory*, hep-th/9904207.

[23] H.B. Nielsen and P. Olesen, *Vortex line models for dual strings*, *Nucl. Phys.* **B 61** (1973) 45.

[24] E. Bogomolnyi, *Stability of classical solutions*, *Sov. J. Nucl. Phys.* **24** (1976) 449.

[25] D. Quillen, *Superconnections and the Chern character*, *Topology* **24** (1985) 89.

[26] E. Moreno, C. Nunez and F.A. Schaposnik, *Electrically charged vortex solution in Born-Infeld theory*, *Phys. Rev.* **D 58** (1998) 025015 [hep-th/9802175].

[27] C. Kennedy and A. Wilkins, *Ramond-Ramond couplings on brane-antibrane systems*, *Phys. Lett.* **B 464** (1999) 206 [hep-th/9905195].

[28] A.P. Polychronakos, *Flux tube solutions in noncommutative gauge theories*, hep-th/0007043.